PERIODIC ATOMIC QUASIINTERPOLATION

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We consider the approximation of periodic functions by using the atomic quasiinterpolation of
the second and the first order. We obtain expressions for the coefficients of quasiinterpolants
and present estimates for errors in the uniform metric.

Let the values of a periodic function \( f(x_j) = f_j \), where \( f(x) \in \mathcal{C}^{m+1}[-\pi, \pi] \) and \( f^{(i)}(-\pi) = f^{(i)}(\pi) \),
\( i = 0, \ldots, m \), be given at the nodes of a uniform grid \( x_j = jh \), \( j = -N, N \), \( h = \pi/N \). Consider an atomic
interpolant of the second order [1–3]

\[
F_2(f; x) = \sum_{j=\frac{-N+1}{2}}^{\frac{N+1}{2}} c_j \up_2\left( \frac{x + \pi}{h} - j \right)
\]

such that \( F_2(f; x_j) = f_j, j = \frac{-N+1}{2}, \frac{N+1}{2} \). Here,

\[
\up_2(x) = B_1(x) \ast \up(x) = B_2(x) \ast \up(2x),
\]

where \( B_n(x) \) is the Schoenberg \( B \)-spline of order \( n \), \( \up(x) \) is a finite solution of the functional differential
equation

\[
y'(x) = 2y(2x+1) - 2y(2x-1), \quad \text{supp } y(x) = [-1,1], \quad y(x) \in \mathcal{C}^\infty(-\infty, \infty),
\]

and \( \ast \) denotes convolution. [In [1], instead of \( \up_n(x) \), the functions \( \operatorname{Fup}_n(x) = \up_n(2^n x) \) were defined.] Since \( \up_2(x) \) can be represented as a linear combination of \( \up(x) \), namely,

\[
\up_2(x) = \begin{cases}
\quad \up\left( \frac{x}{4} - \frac{1}{2} \right) - 2\up\left( \frac{x}{4} - \frac{3}{4} \right) + 2\up\left( \frac{x}{4} - 1 \right) - 2\up\left( \frac{x}{4} - \frac{5}{4} \right), & x \in [-2; 2], \\
\quad 0, & x \not\in [-2; 2],
\end{cases}
\]

interpolant (1) can be rewritten in the following equivalent form:
\[ F_2(f; x) = \sum_j \hat{c}_j \text{up} \left( \frac{x + \pi}{4h} - \frac{j}{4} \right), \] (3)

It follows from (2) that

\[ \text{fup}_2(0) = \frac{26}{72}, \quad \text{fup}_2(\pm 1) = \frac{5}{72}. \] (4)

For the determination of the unknown coefficients \( c_j \) in (1), by using the conditions of interpolation and periodicity and relations (4) we obtain the following system of second-order difference equations:

\[ c_{-N-1} = c_{N-1}, \quad c_{-N+1} = c_{N+1}, \]

\[ F_2(f; x_j) = \frac{5}{72} c_{j-1} + \frac{26}{72} c_j + \frac{5}{72} c_{j+1} = f_j, \quad j = -N, N. \] (5)

It is obvious that

\[ \frac{5}{72} c_{j-1} + \frac{26}{72} c_j + \frac{5}{72} c_{j+1} = \frac{1}{2} c_j + \frac{5}{72} \Delta^2 c_j, \quad j = -N - 1, N + 1, \]

and a solution of system (5) has the form

\[ c_j = 2 \sum_{\nu=0}^{\infty} \left( -\frac{5}{36} \right)^\nu \Delta^{2\nu} f_j, \quad j = -N - 1, N + 1, \] (6)

where \( \Delta^{2\nu} \) is the central divided difference of order \( 2\nu \). Indeed, in this case, we have

\[ F_2(f; x_j) = \frac{1}{2} c_j + \frac{5}{72} \Delta^2 c_j = \sum_{\nu=0}^{\infty} \left( -\frac{5}{36} \right)^\nu \Delta^{2\nu} f_j + \frac{5}{36} \Delta^2 \sum_{\nu=0}^{\infty} \left( -\frac{5}{36} \right)^\nu \Delta^{2\nu} f_j \]

\[ = \sum_{\nu=0}^{\infty} \left( -\frac{5}{36} \right)^\nu \Delta^{2\nu} f_j - \sum_{\nu=0}^{\infty} \left( -\frac{5}{36} \right)^{\nu+1} \Delta^{2\nu+2} f_j \]

\[ = \sum_{\nu=0}^{\infty} \left( -\frac{5}{36} \right)^\nu \Delta^{2\nu} f_j - \sum_{\nu=1}^{\infty} \left( -\frac{5}{36} \right)^\nu \Delta^{2\nu} f_j = f_j. \]

It is obvious that, for the determination of the coefficients \( c_j \), it is necessary to use information about the behavior of the approximated function in the entire domain of its definition. At the same time, as is known [4 – 6], for the spline interpolation of minimum defect one can construct so-called local splines, which enables one to avoid this complication. In the case considered, if we take finitely many terms in series (6), then \( F_2(f; x) \) turns into a quasiinterpolant [4] for the function \( f(x) \). Let us estimate the error of quasiinterpolation.
\[ \left| \Delta^2 v f_j \right| \leq 4^{v-p} \max_k \left| \Delta^2 p f_k \right|, \quad v \geq p, \]

we have

\[ \left| 2 \sum_{v=p}^{\infty} \left( \frac{-5}{36} \right)^v \Delta^2 v f_j \right| \leq \left| 2 \sum_{v=p}^{\infty} \left( \frac{-5}{36} \right)^v 4^{v-p} \max_k \Delta^2 p f_k \right| \leq \frac{2}{4^p} \max_k \left| \Delta^2 p f_k \right| \sum_{v=p}^{\infty} \left( \frac{5}{9} \right)^v \]

\[ \leq \frac{18}{4^{p+1}} \left( \frac{5}{9} \right)^{p-1} \max_k \left| \Delta^2 p f_k \right| = \frac{81}{10} \left( \frac{5}{36} \right)^p \max_k \left| \Delta^2 p f_k \right| = O(h^{2p}) \quad (7) \]

for \( f(x) \in C^{2p} \). Denote

\[ c_{j,p} = 2 \sum_{v=0}^{p-1} \left( \frac{-5}{36} \right)^v \Delta^2 v f_j, \quad j = -N-1, N+1, \]

\[ F_{2,p}(f; x) = 2 \sum_{j=-N-1}^{N+1} c_{j,p} \text{fup}_{2} \left( \frac{x + \pi}{h} - j \right), \quad p = 1, 2, \ldots. \quad (8) \]

In particular, for \( p = 1, 2, \) we get

\[ c_{j,1} = 2 f_j, \quad F_{2,1}(f; x) - f_j = \frac{5}{36} \Delta^2 f_j, \quad j = -N-1, N+1, \]

\[ c_{j,2} = 2 f_j - \frac{5}{18} \Delta^2 f_j, \quad F_{2,2}(f; x) - f_j = -\frac{25}{1296} \Delta^4 f_j, \quad j = -N-1, N+1. \quad (10) \]

The following statement is true:

**Theorem 1.** Let \( f(x) \in C^{m+1}[-\pi, \pi], \) where \( m \geq 2. \) Then

\[ \left\| f(x) - F_{2,1}(f; x) \right\|_{C[-\pi, \pi]} = O(h^2), \quad (11) \]

\[ \left\| f(x) - F_{2,2}(f; x) \right\|_{C[-\pi, \pi]} = O(h^3). \quad (12) \]

**Proof.** By using the results presented in [1–3], we obtain

\[ \left\| (f(x) - F_2(f; x))^{(\alpha)} \right\|_{C[-\pi, \pi]} = O(h^{3-\alpha}), \quad \alpha = 0, 1, 2. \quad (13) \]

Therefore, for \( \alpha = 0, \) we get
\[
\| f(x) - F_{2,p}(f;x) \|_{C[-\pi,\pi]} \leq \| F_2(f;x) - F_{2,p}(f;x) \|_{C[-\pi,\pi]} + \| f(x) - F_2(f;x) \|_{C[-\pi,\pi]}
\]
\[
\leq 2 \sum_{j=1-N-1}^{N+1} \left( \sum_{v=p}^{\infty} \left( \frac{5}{36} \right)^v \Delta^2 f_j \right) \sup_{x} \left( \frac{x+\pi}{h} - j \right) + O(h^3).
\]

On the segment \([-\pi, \pi]\), we have
\[
\sum_{j=1-N-1}^{N+1} \sup_{x} \left( \frac{x+\pi}{h} - j \right) = \frac{1}{2}.
\]

By using (7), we get
\[
\| f(x) - F_{2,p}(f;x) \|_{C[-\pi,\pi]} \leq 2 \max_{k} \left( \sum_{v=p}^{\infty} \left( \frac{5}{36} \right)^v \Delta^2 f_k \right) \sup_{x} \left( \frac{x+\pi}{h} - j \right) + O(h^3)
\]
\[
\leq \frac{81}{10} \left( \frac{5}{36} \right)^p \max_{k} |\Delta^2 f_k| + O(h^3) = O(h^{2p}) + O(h^3),
\]

which yields (11) and (12) for \( p = 1 \) and \( p = 2 \), respectively.

Consider the problem of approximation of derivatives of \( f(x) \). First, note that
\[
F'_2(f;x) = \frac{1}{4h} (c_{j+1} - c_{j-1}), \quad F''_2(f;x) = \frac{1}{2h^2} (c_{j+1} - 2c_j + c_{j-1}),
\]
\[
F^{(n)}_2(f;x) = 0, \quad n > 2, \quad j = -N+1, N+1.
\]

Assuming that \( f(x) \) is fairly smooth and taking relations (9) and (10) into account, at the nodes \( x_j \) we get
\[
F'_{2,1}(f;x) = \frac{1}{4h} (c_{j+1,1} - c_{j-1,1}) = \frac{1}{2h} (f_{j+1} - f_{j-1}) = f'_j + O(h^2),
\]
\[
F''_{2,1}(f;x) = \frac{1}{2h^2} (c_{j+1,1} - 2c_{j,1} + c_{j-1,1}) = \frac{1}{h^2} \Delta^2 f_j = f''_j + O(h^2),
\]
\[
F'_{2,2}(f;x) = \frac{1}{4h} (c_{j+1,2} - c_{j-1,2}) = \frac{1}{2h} (f_{j+1} - f_{j-1}) + \frac{5}{36h} \left( \Delta^2 f_j - \frac{1}{2} \Delta^4 f_j \right) = f'_j + O(h^2),
\]
\[
F''_{2,2}(f;x) = \frac{1}{2h^2} (c_{j+1,2} - 2c_{j,2} + c_{j-1,2}) = \frac{1}{h^2} \Delta^2 f_j - \frac{5}{36h^2} \Delta^4 f_j = f''_j + O(h^2), \quad j = -N+1, N+1.
\]
We use a representation of the form (3) and analogs of the Markov–Bernshtein inequalities for the derivatives of linear combinations of translations of contractions of the functions $u_p(x)$ [1]:

$$\| F'_e(f; x) \|_{C[-\pi, \pi]} \leq 12 \sqrt{\frac{2}{13}} \| F_1(f; x) \|_{C[-\pi, \pi]}.$$  

$$\| F''_e(f; x) \|_{C[-\pi, \pi]} \leq 24 \left( \frac{2}{13} \right)^3 \| F_1(f; x) \|_{C[-\pi, \pi]}.$$  

As in the proof of Theorem 1, we set $\alpha = 1, 2$ in (13) and obtain

$$\left\| (f(x) - F_{2, p}(f; x))^{(\alpha)} \right\|_{C[-\pi, \pi]} = O(h^{3-\alpha}), \quad \alpha = 1, 2, \quad p = 1, 2.$$  

We now pass to a periodic atomic interpolation of the first order. According to [1], it must be represented as follows:

$$F_1(f; x) = \sum_{j=-N-1}^{N} d_j u_p \left( \frac{x + \pi}{h} - j - \frac{1}{2} \right).$$  

(14)

In the case considered, we have $F_1(f; x_{j+1/2}) = f_{j+1/2}, j = -N-1, N$. Here, $x_{j+1/2} = x_j + 0.5h$ and $f_{j+1/2} = f(x_{j+1/2})$. For $u_p(x)$, we get

$$u_p(x) = B_0(x) * u(x),$$

$$u_p(x) = \begin{cases} 
  u_p(x) \left( \frac{x}{2} - \frac{1}{4} \right) - u_p(x) \left( \frac{x}{2} - \frac{3}{4} \right) - u_p(x) \left( \frac{x}{2} - \frac{5}{4} \right), & x \in \left[ -\frac{3}{2}, \frac{3}{2} \right], \\
  0, & x \not\in \left[ -\frac{3}{2}, \frac{3}{2} \right]. 
\end{cases}$$

$$u_p(0) = \frac{62}{72}, \quad u_p(\pm 1) = \frac{5}{72}.$$  

For the determination of the coefficients $d_j$, we obtain the system

$$d_{-N-1} = d_{N-2}, \quad d_{-N+1} = d_N.$$  

$$F_1(f; x_{j+1/2}) = \frac{5}{72} d_{j-1} + \frac{62}{72} d_{j} + \frac{5}{72} d_{j+1} = f_{j+1/2}, \quad j = -N, N-1,$$

whose solution can be represented in the form
\[ d_j = \sum_{\nu=0}^{\infty} \left( \frac{-5}{72} \right)^\nu \Delta^\nu f_{j+1/2}, \quad j = -N-1, N. \]

By analogy with (8), we define
\[ d_{j,p} = \sum_{\nu=0}^{p-2} \left( \frac{-5}{72} \right)^\nu \Delta^\nu f_{j+1/2}, \quad j = -N-1, N, \]

\[ F_{i,p}(f; x) = \sum_{j=-N-1}^{N} d_{j,p} f_{j+1/2} \left( \frac{x+\pi}{h} - j - \frac{1}{2} \right), \quad p = 1, 2, \ldots. \] (15)

It is easy to verify that
\[ d_{j,1} = f_{j+1/2}, \quad F_{1,1}(f; x_{j+1/2}) = f_{j+1/2} + \frac{5}{72} \Delta^2 f_{j+1/2} = f_{j+1/2} + O(h^2), \]
\[ F'_1(f; x_{j+1/2}) = \frac{1}{2h} (d_{j+1}-d_{j-1}), \]
\[ F'_{1,1}(f; x_{j+1/2}) = \frac{1}{2h} (d_{j+1,1}-d_{j-1,1}) = \frac{1}{2h} (f_{j+3/2}-f_{j-1/2}) = f'_{j+1/2} + O(h^2) \]

for \( f(x) \in C^2[-\pi, \pi] \). By virtue of the fact that [1–3]
\[ \left\| (f(x) - F_1(f; x))^{(\alpha)} \right\|_{C[-\pi, \pi]} = O(h^{2-\alpha}), \quad \alpha = 0, 1, \]
the following statement is true:

**Theorem 2.** Let \( f(x) \in \tilde{C}^{m+1}[-\pi, \pi] \), where \( m \geq 1 \). Then
\[ \left\| (f(x) - F_{1,1}(f; x))^{(\alpha)} \right\|_{C[-\pi, \pi]} = O(h^{2-\alpha}), \quad \alpha = 0, 1. \]

Thus, the error of quasiinterpolation of the function \( f(x) \) and its derivatives with the use of \( F_{1,1}(f; x) \) and \( F_{2,2}(f; x) \) is of the same order as the error of the atomic interpolation by the expressions \( F_1(f; x) \) and \( F_2(f; x) \), respectively. However, in contrast to (1) and (14), the coefficients in relations (8) and (15) possess the property of locality. Analogous results were obtained earlier for parabolic and cubic \( B \)-splines, (see, e.g., [7, 8]).

**REFERENCES**